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GLOBAL CONTROLLABILITY OF SAMPLED-DATA BILINEAR TIME-DELAY SYSTEMS†

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The concept of the universal control of a controllable sampled-data bilinear time-delay system is introduced. A universal control is independent of the initial state, and the system may be steered from any initial state at time t_0 to zero at the time t_1. A criterion of global controllability is obtained. As an example, the control of a two-link oscillatory system is considered.

Problems of the control dynamical objects using sampled-data control have many applications. For a survey of the main publications on sampled-data systems see [1, 2].‡

1. DEFINITION OF THE SOLUTION OF A SAMPLED-DATA LINEAR TIME-DELAY SYSTEM

Definition 1.1. A sampled-data linear time-delay system (a sampled-data system) is defined to be the following expression

$$\dot{x} = A(t)x + \sum_{i=1}^{k} \delta(t-\tau_i)H_i(t)x(t-0) \tag{1.1}$$

where $x(\cdot) : R \to \mathbb{C}^n$ (or $\mathbb{R}^n$), $A(t)$ and $H_i(t)$ are square matrices of order n with continuous complex-valued or real-valued elements, $\delta(\cdot)$ is the delta function, and $\tau_1 \le \tau_2 \le \ldots \le \tau_k$ are the data points.

The following initial condition is specified at the point t_0

$$x(t_0) = x_0, \quad \text{where} \quad t_0 = \tau_0 < \tau_1 \tag{1.2}$$

Let $X(t, s)$ denote the Cauchy matrix of the system $\dot{x} = A(t)x$. We define the influence matrix of the ith pulse as the matrix $E + H_i(\tau_i)$. Intuitively, this means that if x_0 is the value of some solution of system (1.1) "before" the ith pulse, then $(E + H_i(\tau_i))x_0$ is the value of the solution "after" the ith pulse. Then the solution of system (1.1) satisfying the initial condition (1.2) will have the form

$$x(t) = X(t, \tau_{k(t)})\prod_{i=1}^{k(t)} [(E + H_i(\tau_i))X(\tau_i, \tau_{i-1})]x_0$$

where $k(t)$ is the maximum subscript i such that $\tau_i < t$. Henceforth the product symbol is understood in the sense of left matrix multiplication, that is, $\prod_{i=1}^{k} A_i = A_k A_{k-1} \ldots A_1$. Using the Heaviside function,

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‡See also SESEKIN, A. A., Dynamical systems with non-linear sampled-data structure. Doctorate Dissertation, 01.01.02. Inst. Mat. Mekh., Ural'sk Otd. Ross. Akad. Nauk, Ekaterinburg, 1997.

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one can eliminate the function $k(t)$ and give the following equivalent definition of the solution of a sampled-data system.

Definition 1.2 A solution of the sampled-data Cauchy problem (1.1), (1.2) is a function

$$x(t) = X(t, \tau_k)\prod_{i=1}^{k}[(E+\chi(t-t_i)H_i(\tau_i))X(\tau_i, \tau_{i-1})]x_0 \tag{1.3}$$

where $\chi(\cdot)$ is the Heaviside function: $\chi(t) = 0$ for $t < 0$, $\chi(t) = 1$ for $t > 0$.

At the points τ_i the function $x(\cdot)$ is undefined (if necessary, it may be defined be left or right continuity). It is important to note that the definition of a sampled-data system and its solution explicitly involves the numbering of the points τ_i, which reflects the order of the sequence of pulses. In that connection, points τ_i cannot be interchanged even if they coincide, since the product of the corresponding matrices $E + H_i(\tau_i)$ is generally non-commutative. This means that a change in the order of the pulses concentrated at one data point may change the solution of the system.

Consider a family of systems

$$\dot{x} = A(t)x + \sum_{i=1}^{k}\delta_i(t-\tilde{\tau}_i)H_i(t)x(\tilde{\tau}_i-\varepsilon_2) \tag{1.4}$$

which depend on the numbers $\varepsilon_1, \varepsilon_2, \varepsilon_3$, instants of time $\tilde{\tau}_i$ and functions $\delta_i(\cdot)$, and satisfying the following approximation conditions: (1) the functions $\delta_i(\cdot)$ are continuous throughout $(-\infty, \infty)$; $\delta_i(t) \geq 0$ for all t; $\delta_i(t) = 0$ for all $t \notin (-\varepsilon_1, \varepsilon_1)$, and $\int_{-\varepsilon_1}^{\varepsilon_1}\delta_i(t)dt = 1$; (2) $\varepsilon_2 > \varepsilon_1 > 0$; (3) $|\tilde{\tau}_i - \tau_i| \leq \varepsilon_3$ for all $i = 1, \ldots, k$; (4) $|\tilde{\tau}_{i+1} - \tilde{\tau}_i| > \varepsilon_1 + \varepsilon_2$ for all $i - 0, \ldots, k-1$.

Condition 1 describes the approximation of a delta function with pulse half-width ε_1. Condition 2 means that the value of the solution is measured at a time $\tilde{\tau}_i - \varepsilon_2$, and then the pulse in the interval $[\tilde{\tau}_i - \varepsilon_1, \tilde{\tau}_i + \varepsilon_1]$ is produced on the basis of the measured values, except that the delay ε_2 exceeds the pulse half-width ε_1. The third condition introduces an estimate of the closeness of the points $\tilde{\tau}_i$ and τ_i. The fourth condition means that the next value of the solution is measured after completion of the previous pulse.

Over the interval $[t_0, \tilde{\tau}_1 - \varepsilon_1]$ all the functions $\delta_i(\cdot)$ vanish, and therefore solutions of system (1.4) are understood in the classical sense and are identical with the solutions of the system $\dot{x} = A(t)x$. Moreover, the value of the solution $x(\tilde{\tau}_1 - \varepsilon_2)$ has already been defined, so that in the interval $[\tilde{\tau}_1 - \varepsilon_1, \tilde{\tau}_1 + \varepsilon_1]$ the solutions of system (1.4) are also understood in the classical sense. Then, proceeding in a similar way, the solutions are defined over the interval $[\tilde{\tau}_1 + \varepsilon_1, \tilde{\tau}_2 - \varepsilon_1]$, over the interval $[\tilde{\tau}_2 - \varepsilon_1, \tilde{\tau}_2 + \varepsilon_1]$, and so on.

Definition 1.3. The solutions of system (1.1) are approximated by solutions of system (1.4) uniformly on a set I if, for any arbitrarily small $\varepsilon > 0$ and any vector x_0, numbers r_1, r_2, r_3 and r_4 exist such that, if $|\varepsilon_1| \leq r_1$, $|\varepsilon_2| \leq r_2$, $|\varepsilon_3| \leq r_3$, $|\tilde{x}_0 - x_0| \leq r_4$ and the approximation conditions hold, then $|\tilde{x}(t) - x(t)| < \varepsilon$ for all t in the set I. Here $\tilde{x}(\cdot)$ is the solution of system (1.4) with initial condition $x(t_0) = \tilde{x}_0$, and $x(\cdot)$ is the solution of the Cauchy sampled-data system (1.1), (1.2).

Theorem 1.1. For any $r > 0$ and $T > t_0$, the solution of system (1.1) is approximated by solutions of system (1.4) uniformly in the set

$$I = [t_0, T]\backslash\bigcup_{i=1}^{m}(\tau_i - r, \tau_i + r)$$

Proof. Fix arbitrary $r > 0$ and $T > t_0$. Let $\tilde{x}(\cdot)$ be the solution of system (1.4) with initial condition $x(t_0) = \tilde{x}_0$ and $x(\cdot)$ the solution of the sampled-data Cauchy system (1.1), (1.2) defined by formula (1.3). For sufficiently small ε_1, ε_2 and ε_3, the inclusion relation $[\tilde{\tau}_i - \varepsilon_2, \tilde{\tau}_i + \varepsilon_1] \subseteq (\tau_i - r, \tau_i + r)$ holds. Then, by the Cauchy formula, for all $t \in I$.

$$\tilde{x}(t) = X(t, \tilde{\tau}_k + \varepsilon_1)\prod_{i=1}^{k}[(X(\tilde{\tau}_i + \varepsilon_1, \tilde{\tau}_i - \varepsilon_2) + \chi(t-\tau_i)B_i)X(\tilde{\tau}_i - \varepsilon_2, \tilde{\tau}_{i-1} + \varepsilon_1)]X(t_0 + \varepsilon_1, t_0)\tilde{x}_0 \tag{1.5}$$

where

$$B_i = \int_{\tilde{\tau}_i - \varepsilon_2}^{\tilde{\tau}_i + \varepsilon_1} \delta_i(s - \tilde{\tau}_i) X(\tilde{\tau}_i + \varepsilon_1, s) H_i(s) ds$$

Since the Cauchy matrix is continuous, the following relations holds as $\varepsilon_1 \to 0$, $\varepsilon_2 \to 0$, $\varepsilon_3 \to 0$ and $\tilde{\tau}_i \to \tau_i$

$$X(t, \tilde{\tau}_k + \varepsilon_1) \to X(t, \tau_k), \quad X(\tilde{\tau}_i + \varepsilon_1, \tilde{\tau}_i - \varepsilon_2) \to E$$

$$X(\tilde{\tau}_i - \varepsilon_2, \tilde{\tau}_{i-1} + \varepsilon_1) \to X(\tau_i, \tau_{i-1}), \quad X(t_0 + \varepsilon_1, t_0) \to E$$

and the convergence is uniform to t on the set I.

Applying the mean-value theorem for integrals to each element of the matrix B_i, we obtain

$$B_i = P(\bar{\xi}) \int_{\tilde{\tau}_i - \varepsilon_2}^{\tilde{\tau}_i + \varepsilon_1} \delta_i(s - \tilde{\tau}_i) ds = P(\bar{\xi})$$

where $P(\bar{\xi})$ is a matrix whose elements are the values of the elements of the matrices $X(\tilde{\tau}_i + \varepsilon_1, s)H_i(s)$ at certain points $\xi_{j,k} \in [\tilde{\tau}_i - \varepsilon_2, \tilde{\tau}_i + \varepsilon_1]$. Then, since the matrices X and H are continuous, we conclude that the matrix $P(\bar{\xi})$ tends uniformly to $H_i(\tau_i)$ as $\varepsilon_1 \to 0$, $\varepsilon_2 \to 0$, $\varepsilon_3 \to 0$ and $\tilde{\tau}_i \to \tau_i$. Consequently, $\tilde{x}(t)$ converges uniformly on I to $x(t)$, which proves the theorem.

An analogous theorem was proved in [3, 4] using the technique of non-standard analysis.

2. SAMPLED-DATA BILINEAR TIME-DELAY CONTROLLABLE SYSTEMS

We shall consider a controllable sampled-data bilinear time-delay system (controllable sampled-data system) over the interval $I = [t_0, t_1]$

$$\dot{x}(t) = A(t)x(t) + B(t)U(t)x(t-0) \tag{2.1}$$

where $A(t)$ and $B(t) = (b_1(t), \dots, b_m(t))$ are $n \times n$ and $n \times m$ matrices with continuous elements, and $U(\cdot)$ belongs to the set of admissible controls (see the next definition).

Definition 2.1. An admissible control $U(\cdot)$ is defined as any finite sequence of pairs $\{(\tau_k, U_k)\}_{k=1}^{p}$ such that U_k are $m \times n$ matrices and $t_0 = \tau_0 < \tau_1 \le \tau_2 \le \dots \le \tau_p < t_1$. In that situation we use the formal notation

$$U(t) = \sum_{k=1}^{p} \delta(t - \tau_k) U_k \tag{2.2}$$

Definition 2.2. A solution of the sampled-data Cauchy system (2.1), (1.2) is, by Definition 1.2, a function

$$x(t, x_0, U) = X(t, \tau_p) \prod_{k=1}^{p} [(E + \chi(t - \tau_k) B(\tau_k) U_k) X(\tau_k, \tau_{k-1})] x_0$$

Let

$$x(\tau_j - 0, x_0, U) = X(\tau_j, \tau_{j-1}) \prod_{k=1}^{j-1} [(E + B(\tau_k) U_k) X(\tau_k, \tau_{k-1})] x_0$$

denote the value of the solution $x(\cdot, x_0, U)$ "before" the pulse concentrated at the point τ_j. Let

$$x(\tau_j + 0, x_0, U) = (E + B(\tau_j) U_j) x(\tau_j - 0, x_0, U)$$

denote the value of the solution $x(\cdot, x_0, U)$ "after" the pulse concentrated at the point τ_j. In the case when all the τ_k are different, these are simply the left and right limits of the function $x(\cdot, x_0, U)$ at the point τ_j.

3. THE GLOBAL CONTROLLABILITY SET

Definition 3.1. We define the controllability set of system (2.1) on I to be the set of all vectors x_0 such that $x(t_1, x_0, U) = 0$ for some admissible control U.

Given system (2.1), we construct the sets

$$M_j = \sum_{s \in (t_0, t_1)} \langle X(t_0, s) b_j(s) \rangle, \quad M = \sum_{j=1}^{m} M_j \tag{3.1}$$

where the angular brackets denote the linear span of the vector and the summation symbol denotes the sum of linear subspaces, understood in the following sense: $h \in M_j$ if and only if a finite number of points $\tau_1, \dots, \tau_q$, subscripts $j_1, \dots, j_q$ and vectors $h_1, \dots, h_q$ exist such that $t_k \in (t_0, t_1)$, $h = h_1 + \dots + h_q$, where $h_k \in \langle X(t_0, \tau_k) b_{j_k}(\tau_k) \rangle$.

Lemma 3.1. The controllability set of system (2.1) on I is subset of M.

Proof. Let x_0 be an arbitrary vector such that $x(t_1, x_0, U) = 0$ for some admissible control $U(t)$ (2.2). For any matrix U_k we have the representation

$$(E + B(\tau_k) U_k) x_0 = x_0 + \sum_{j=1}^{m} c_{k,j} b_j(\tau_k), \quad \mathrm{col}(c_{k,1}, \dots, c_{k,m}) = U_k x_0$$

Then

$$x(\tau_1 - 0, x_0, U) = X(\tau_1, t_0) x_0$$

$$x(\tau_1 + 0, x_0, U) = (E + B(\tau_1) U_1) x(\tau_1 - 0, x_0, U) = X(\tau_1, t_0) x_0 + \sum_{j=1}^{m} c_{1,j} b_j(\tau_1)$$

$$x(\tau_2 - 0, x_0, U) = X(\tau_2, \tau_1) x(\tau_1 + 0, x_0, U)$$

$$x(\tau_2 + 0, x_0, U) = X(\tau_2, t_0) x_0 + \sum_{j=1}^{m} [c_{1,j} X(\tau_2, \tau_1) b_j(\tau_1) + c_{2,j} b_j(\tau_2)]$$

Continuing in the same way, we find by induction that

$$x(\tau_p + 0, x_0, U) = X(\tau_p, t_0) x_0 + \sum_{k=1}^{p} \sum_{j=1}^{m} c_{k,j} X(\tau_p, \tau_k) b_j(\tau_k)$$

$$x(t_1, x_0, U) = X(t_1, t_0) x_0 + \sum_{k=1}^{p} \sum_{j=1}^{m} c_{k,j} X(t_1, \tau_k) b_j(\tau_k) = 0$$

Multiplying the last equality on the left by $X(t_0, t_1)$, we get

$$x_0 = -\sum_{k=1}^{m} \sum_{j=1}^{p} c_{k,j} X(t_0, \tau_k) b_j(\tau_k)$$

Consequently, $x_0 \in M$.

Lemma 3.2. An admissible control U exists such that $x(t_1, x_0, U) = 0$ for all $x_0 \in M$.

Proof. Since $M \subseteq \mathbb{R}^n$, it follows that a number $p \le n$, points $\tau_1, \dots, \tau_p$ in $[t_0, t_1]$, and subscripts $j_1, \dots, j_p$ exist such that $M = \sum_{i=1}^{p} \langle X(t_0, \tau_i) b_{j_i}(\tau_i) \rangle$; in addition, the system of vectors $\{X(t_0, \tau_i) b_{j_i}(\tau_i)\}_{i=1}^{p}$ is linearly independent.

Consider the control

$$U(t) = \sum_{k=1}^{p} \delta(\tau_k) U_k$$

in which all the rows of the matrices U_k except the j_kth consist of zeros, while the j_kth row is $(\alpha_{k,1}, \dots, \alpha_{k,n})$, where $\alpha_{k,j}$ are numbers satisfying the linear system of equations

$$(\alpha_{k,1}, \dots, \alpha_{k,n})(X(\tau_k, \tau_k) b_{j_k}(\tau_k), \dots, X(\tau_k, \tau_p) b_{j_p}(\tau_p)) = (-1, 0, \dots, 0) \tag{3.2}$$

Since the system of vectors $\{X(t_0, \tau_i) b_{j_i}(\tau_i)\}_{i=1}^{p}$ is linearly independent, the system $\{X(\tau_k, \tau_i) b_{j_i}(\tau_i)\}_{i=k}^{p}$ is also linearly independent. Consequently, system (3.2) is solvable, that is, a control U exists (though it need not be unique).

By the construction of the control U, we obtain

$$B(\tau_k) U_k = b_{j_k}(\tau_k)(\alpha_{k,1}, \dots, \alpha_{k,n})$$

$$(\alpha_{k,1}, \dots, \alpha_{k,n}) X(\tau_k, \tau_i) b_{j_i}(\tau_i) = \begin{cases} -1, & \text{if} \quad i = k \\ 0, & \text{if} \quad i > k \end{cases}$$

Then

$$(E + B(\tau_k) U_k) X(\tau_k, \tau_i) b_{j_i}(\tau_i) = \begin{cases} 0, & \text{if} \quad i = k \\ X(\tau_k, \tau_i) b_{j_i}(\tau_i), & \text{if} \quad i > k \end{cases} \tag{3.3}$$

Let $x_0 \in M$ be an arbitrary vector. Then, using relation (3.3), we get

$$x(\tau_1 - 0, x_0, U) \in \sum_{k=1}^{p} \langle X(\tau_1, \tau_k) b_{j_k}(\tau_k) \rangle$$

$$x(\tau_1 + 0, x_0, U) = (E + B(\tau_1) U_1) x(\tau_1 - 0, x_0, U) \in \sum_{k=2}^{p} \langle X(\tau_1, \tau_k) b_{j_k}(\tau_k) \rangle$$

$$x(\tau_2 - 0, x_0, U) = X(\tau_2, \tau_1) x(\tau_1 + 0, x_0, U) \in \sum_{k=2}^{p} \langle X(\tau_2, \tau_k) b_{j_k}(\tau_k) \rangle$$

$$x(\tau_2 + 0, x_0, U) = (E + B(\tau_2) U_2) x(\tau_2 - 0, x_0, U) \in \sum_{k=3}^{p} \langle X(\tau_2, \tau_k) b_{j_k}(\tau_k) \rangle$$

$$\dots$$

$$x(\tau_p - 0, x_0, U) \in \langle X(\tau_p, \tau_p) b_{j_p}(\tau_p) \rangle = \langle b_{j_p}(\tau_p) \rangle$$

$$x(\tau_p + 0, x_0, U) = (E + B(\tau_p) U_p) x(\tau_p - 0, x_0, U) \in \{0\}$$

$$x(t_1, x_0, U) = X(t_1, \tau_p) x(t_p + 0, x_0, U) = 0$$

Since x_0 was an arbitrary vector, this proves the lemma.

Theorem 3.1. The set M is the controllability set of system (2.1) on I. Moreover, an admissible control U exists such that $x(t_1, x_0, U) = 0$ for all $x_0 \in M$.

The assertion of Theorem 3.1 follows from the two preceding lemmas.

Theorem 3.1 implies that in sampled-data systems, unlike the classical case, a single admissible control U exists that will steer the system from any initial state in M at time t_0 to zero at time t_1, and moreover neither the times τ_k nor the matrices U_k occurring in U depend on the initial state x_0. In order to emphasize this fact, we shall call the controllability set the global controllability set, and the control U in the assertion of Theorem 3.1 will be called a universal control.

Note that the proof of Theorem 3.1 readily implies a constructive way of obtaining a universal control U.

Definition 3.2. System (2.1) is said to be globally controllable on I if its global controllability set is $\mathbb{R}^n$.

Definition 3.3. We shall say that a system is globally controllable over the interval $[t_0 - 0, t_1 + 0]$ if it is globally controllable over any interval $[\tilde{t}_0, \tilde{t}_1]$ such that $\tilde{t}_0 < t_0$ and $\tilde{t}_1 > t_1$.

Consider the matrix

$$D_n = \begin{Vmatrix} f_{1,j_1}(\tau_1) & f_{1,j_2}(\tau_2) & \dots & f_{1,j_n}(\tau_n) \\ f_{2,j_1}(\tau_1) & f_{2,j_2}(\tau_2) & \dots & f_{2,j_n}(\tau_n) \\ \vdots & \vdots & & \vdots \\ f_{n,j_1}(\tau_1) & f_{n,j_2}(\tau_2) & \dots & f_{n,j_n}(\tau_n) \end{Vmatrix}$$

Lemma 3.3 (on linearly independent functions). A necessary and sufficient condition for the functions $f_1(t), f_2(t), \dots, f_n(t)\colon I \to \mathbb{R}^m$ to be linearly independent over the set I is that points $\tau_1, \dots, \tau_n \in I$ and subscripts $j_1, \dots, j_n$ exist such that $\det D_n \neq 0$.

Proof. Sufficiency is obvious. We will prove necessity by induction on n. The assertion is obvious for $n = 1$. Supposing it is true for $n < k$, we prove it for $n = k$. Let the system of functions $\{f_i(t)\}_{i=1}^{k}$ be linearly independent of I. Consequently, by the induction hypothesis, system of points $\{\tau_i\}_{i=1}^{k-1}$ and subscripts $\{j_i\}_{i=1}^{k-1}$ exist such that $\det D_{k-1} \neq 0$. Then the last row of the matrix

$$\begin{Vmatrix} f_{1,j_1} & \dots & f_{1,j_{k-1}}(\tau_{k-1}) \\ \vdots & & \vdots \\ f_{k,j_1} & \dots & f_{k,j_{k-1}}(\tau_{k-1}) \end{Vmatrix}$$

may be expressed uniquely as a linear combination of the preceding rows. Denote the coefficients of this linear combination by $c_1, \dots, c_{k-1}$. Since the functions $f_1, \dots, f_k$ are linearly independent of I, a point $\tau_k \in I$ exists such that

$$f_k(\tau_k) \neq c_1 f_1(\tau_k) + \dots + c_{k-1} f_{k-1}(\tau_k)$$

Then there is a subscript j_k such that

$$f_{k,j_k}(\tau_k) \neq c_1 f_{1,j_k}(\tau_k) + \dots + c_{k-1,j_k} f_{k-1}(\tau_k)$$

Consequently, since the coefficients c_i are unique, the rows of the matrix D_n are linearly independent, and its determinant does not vanish.

Now consider the rows the matrix $X(t_0, s)B(s)$ as functions of the variable s. Taking a maximum linearly independent subsystem of rows in the interval $[t_0, t_1]$, we express $X(t_0, s)B(s)$ as

$$X(t_0, s)B(s) = h_1 f_1(s) + h_2 f_2(s) + \dots + h_q f_q(s)$$

where $h_1, \dots, h_q$ are certain constant vectors and the row-functions $f_1(\cdot), \dots, f_q(\cdot)$ are linearly independent in the interval $[t_0, t_1]$.

Theorem 3.2 The global controllability set M is the linear span $\langle h_1, \dots, h_q\rangle$.

Proof. By formula (3.1) it is obvious that $M \subseteq \langle h_1, \dots, h_q\rangle$. We will prove that $\langle h_1, \dots, h_q\rangle \subseteq M$. Let h be an arbitrary vector such that $h = \alpha_1 h_1 + \dots + \alpha_q h_q$. Since the functions $f_k(\cdot)$ are linearly independent, it follows by Lemma 3.3 that points $s_1, \dots, s_q$ and subscripts $j_1, \dots, j_q$ exist such that the matrix

$$D = \begin{Vmatrix} f_{1,j_1}(s_1) & \dots & f_1(s_{q,j_q}) \\ \vdots & & \vdots \\ f_{q,j_1}(s_1) & \dots & f_q(s_{q,j_q}) \end{Vmatrix}$$

is non-singular. Then

$$X(t_0, s_k) b_{j_k}(s_k) = \sum_{i=1}^{q} h_i f_{i,j_k}(s_k)$$

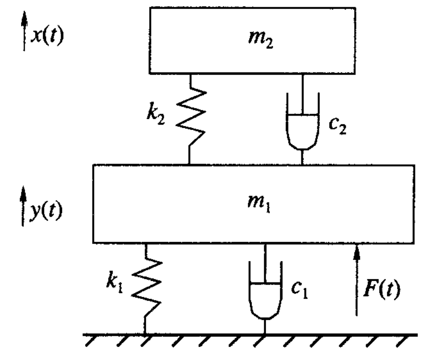

Fig. 1

Equating the coefficients of the vectors h_i, we get

$$h = \sum_{k=1}^{q} c_k X(t_0, s_k) b_{j_k}(s_k)$$

where the constants c_k satisfy the linear system of equations

$$D\mathrm{col}(c_1, \ldots, c_q) = \mathrm{col}(\alpha_1, \ldots, \alpha_q)$$

Consequently, $h \in M_j$.

Theorem 3.3. The sampled-data system (2.1) is globally controllable over the interval $[t_0, t_1]$ if and only if the corresponding classical system $\dot{x} = A(t)x + B(t)u(t)$ is completely controllable over the interval $[t_0, t_1]$.

Proof. It follows from Theorem 3.2 that system (2.1) is globally controllable over I if an only if the rows of the matrix $X(t_0, s)B(s)$ are linearly independent over I. By Krasovskii's criterion [5], linear independence of the rows of the matrix $X(t_0, s)B(s)$ is equivalent to complete controllability of the classical linear control system $\dot{x} = A(t)x + B(t)u(t)$.

Corollary. Over any interval $[t_0, t_1]$ the global controllability set of a stationary sampled-data system.

$$\dot{x}(t) = Ax(t) + BU(t)x(t-0)$$

is the linear span of the columns of the matrices $B, AB, A^2B, \ldots, A^{n-1}B$.

Example. Consider the construction illustrated in Fig. 1, consisting of two oscillating elements. Here m_1 and m_2 denote the masses of the elements, k_1 and k_2 are the stiffnesses, and c_1 and c_2 are the viscosity coefficients.

Let x and y denote the displacements of the masses m_1 and m_2 relative to their equilibrium positions. By Newton's second law, the motion of the construction is described by the following system of equations

$$m_2\ddot{x} = -k_2(x-y) - c_2(\dot{x}-\dot{y})$$

$$m_1\ddot{y} = -k_1 y - c_1\dot{y} + k_2(x-y) + c_2(\dot{x}-\dot{y}) + F(t)$$

Putting $z = \mathrm{col}(x, \dot{x}, y, \dot{y})$ and $u(t) = F(t)/m_1$, we arrive at a system of linear fourth-order differential equations $\dot{z} = Az + u(t)B$, where

$$A = \left\| \begin{matrix} 0 & 1 & 0 & 0 \\ -\frac{k_2}{m_2} & -\frac{c_2}{m_2} & \frac{k_2}{m_2} & \frac{c_2}{m_2} \\ 0 & 0 & 0 & 1 \\ \frac{k_2}{m_1} & \frac{c_2}{m_1} & -\frac{k_1+k_2}{m_1} & -\frac{c_1+c_2}{m_1} \end{matrix} \right\|, \quad B = \left\| \begin{matrix} 0 \\ 0 \\ 0 \\ 1 \end{matrix} \right\|$$

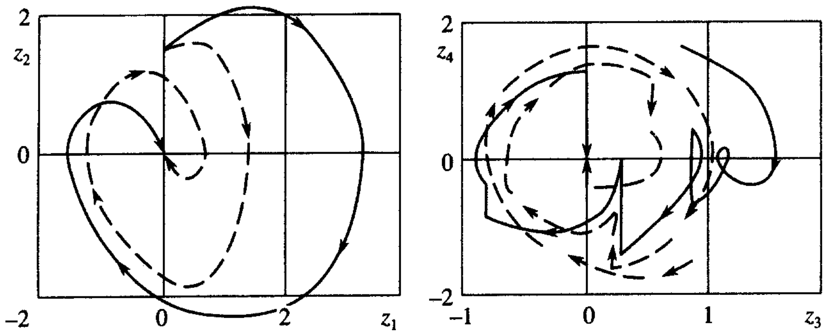

Fig. 2

We construct a control $u(t)$ according to the feedback principle, that is

$$u(t) = U(t)z(t-0)$$

We then obtain the closed-loop system

$$\dot{z}(t) = Az(t) + BU(t)z(t-0) \tag{3.4}$$

Fix the parameter values of the construction as

$$m_1 = 10, \quad m_2 = 7, \quad k_1 = 10, \quad k_2 = 8, \quad c_1 = 0.1, \quad c_2 = 0.2$$

By the Corollary to Theorem 3.3, system (3.4) will be globally controllable over any interval. We fix the initial time and data times as

$$t_0 = 0, \quad \tau_1 = 4, \quad \tau_2 = 5, \quad \tau_3 = 6.5, \quad \tau_4 = 8$$

We construct a universal control (2.2) ($p = 4$) by the method described in the proof of Lemma 3.2. We obtain

$$U_1 \approx (0.4694, -0.3791, -0.6823, -1)$$

$$U_2 \approx (0, 0.6026, 1.0073, -1)$$

$$U_3 \approx (0, 0, 0.4186, -1)$$

$$U_4 \approx (0, 0, 0, -1)$$

In Fig. 2 we show the trajectories of system (3.4) corresponding to the initial condition $z(0) = \mathrm{col}(0, 1.6, 0.8, 1.6)$ (the solid curves), and the trajectories for the initial condition $z(0) = \mathrm{col}(0, 1.6, 0.8, -1.6)$ (the dashed curves).

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